Core Model derived scales and the Approachable Free Subset Property

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Powerset of Singular Cardinals

Let λ be a singular strong limit cardinal. What is 2^λ? (From now on $\lambda = \aleph_\omega$).
Shelah proved $2^{\aleph_\omega} < \aleph_{\omega_4}$. Shelah also hypothesized $2^{\aleph_\omega} < \aleph_{\omega_1}$. (PCF conjecture (simplified)).

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Theorem (Shelah)

If $\aleph_\omega$ is a strong limit but $2^{\aleph_\omega} > \aleph_{\omega_1}$, then the approachable free subset property holds at $\aleph_\omega$. (AFSP($\aleph_\omega$))

Approachable Sets, Free Subsets

Definition

Let $X \prec (H(\theta); \in, \ldots)$, $\kappa < \theta$ an uncountable regular cardinal. X is called internally approachable of length κ, iff there exists $\langle X_\alpha : \alpha < \kappa \rangle$ s.t.

- $X = \bigcup_{\alpha < \kappa} X_\alpha$,
- $X_\alpha \prec (H(\theta); \in, \ldots)$ and $\mathrm{Card}(X_\alpha) < \kappa$ for all $\alpha < \kappa$,
- $\langle X_\beta : \beta < \alpha \rangle \in X_\alpha$ for all $\alpha < \kappa$.

Definition

Let $X \prec (H(\theta); \in, \ldots)$, $\{\kappa_\alpha : \alpha < \lambda\} \subset \theta$ is free over X iff $\chi_X(\kappa_\beta) \notin \mathrm{Sk}^{(H(\theta);\in,\ldots)}(X \cup \{\chi_X(\kappa_\alpha) : \alpha \in a\})$ for all finite $a \subset \lambda$ and $\beta \in \lambda \backslash a$.

Approachable Free Subset Property

Definition

AFSP($\aleph_\omega$) iff for all internally approachable $X \prec (H(\theta); \in, \ldots)$ $(\theta \geq (2^{\aleph_\omega})^+)$ with PCF($\{\aleph_n : n < \omega\}$) $\subset X$ of length $\kappa < \aleph_\omega$ there exists $A \subset \{\aleph_n : n < \omega\}$ infinite that is free over X.

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AFSP($\aleph_\omega$) iff for all internally approachable $X \prec (H(\theta); \in, \dots)$ $(\theta \geq (2^{\aleph_\omega})^+)$ with $\mathrm{PCF}(\{\aleph_n : n < \omega\}) \subset X$ of length $\kappa < \aleph_\omega$ there exists $A \subset \{\aleph_n : n < \omega\}$ infinite that is free over X.

Corollary

AFSP($\aleph_\omega$) implies the existence of an inner model M such that for all $\alpha < \aleph_\omega$ there is $\beta < \aleph_\omega$ with $\mathrm{o}^M(\beta) \geq \alpha$. In fact, there is $\langle U_\gamma : \gamma < \alpha \rangle$ a sequence of normal measures on β with $U_\delta \in \mathrm{Ult}(M; U_\gamma)$ for $\delta < \gamma < \alpha$.

PCF simplified

Basic objects are functions $f \in \prod_{n<\omega} \aleph_n$ ordered by $f <_J g$ iff $\{n < \omega | g(n) \leq f(n)\} \in J$ where J is an ideal on ω. J will usually take the form of $I^A_{<\omega} := \{b \subset \omega | \exists m \forall n > m : n \notin A\}$.
Have $\prod_{n<\omega} \aleph_n / I^A_{<\omega} \cong \prod_{n \notin A} \aleph_n / I_{<\omega} = \prod_{n \in \omega \setminus A} \aleph_n / I_{<\omega}$.

Definition

A sequence $\langle f_\alpha : \alpha < \lambda \rangle$ is a scale (on $\prod_{n<\omega} \aleph_n / J$) iff it is $<_J$-increasing and cofinal, i.e. for all $g \in \prod_{n<\omega} \aleph_n$ there is $\alpha < \lambda$ with $g <_J f_\alpha$.

Tree-like Scales

Definition (Pereira)

A scale $\langle f_\alpha : \alpha < \lambda \rangle$ is tree-like iff $f_\alpha(m) = f_\beta(m) \Rightarrow f_\alpha(n) = f_\beta(n)$ for all $\alpha < \beta < \lambda$ and $n < m < \omega$.

Definition

A scale $\langle f_\alpha : \alpha < \lambda \rangle$ is continuous iff f_α is an exact upper bound for $\langle f_\beta : \beta < \alpha \rangle$, i.e. $f \leq_J g$ for any other upper bound g, for all limit ordinals $\alpha < \lambda$.

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By work of Pereira and later Cummings we do know that continuous tree-like scales can co-exist with strong large cardinal notions, I_0-cardinals and supercompact cardinals respectively. We will show that they exist in V assuming anti-large cardinal assumption.

Anti-AFSP scales

Lemma

Let $\kappa < \aleph_\omega$ *be regular uncountable. Let* $X \prec (H(\theta); \in; \ldots)$ *be internally approachable of length* κ. *Let* $\langle f_\alpha : \alpha < \lambda \rangle \in X$ *be a scale. Then* $\langle \chi_X(\aleph_n) : n < \omega \rangle$ *is an exact upper bound for* $\langle f_\alpha : \alpha < \chi_X(\lambda) \rangle$.

Lemma (Pereira)

Let $A \subset \omega$ *be infinite. Assume* $\prod_{n \in A} \aleph_n / I_{<\omega}$ *carries a continuous tree-like scale, then for a relative club of internally approachable* $X \prec (H(\theta); \in; \ldots)$, A *is not free over* X.

Mice

Let $(*)_\lambda$ be the statement "for all $\alpha < \lambda$ there exists $\beta < \lambda$ with $o(\beta) \geq \alpha$". What would a canonical model of $(*)_\lambda$ M look like?

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Such a model (with some additional properties) is called a *premouse*.

Iterations

Let M be a premouse. An iteration on M is a sequence $\langle M_\alpha, i_{\alpha,\beta} : \alpha \leq \beta \leq \gamma \rangle$ of premice with (partial) embeddings between them, starting with M, produced by iterating the following operations:

- Applying an extender $E \in M_\alpha$ to some M_β ($\beta \leq \alpha$) (for our purposes $\beta = \alpha$ always);
- truncating to an inital segment (but only finitely often);
- taking direct limits (at limit stages).

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Lemma (Comparison)

Let M, N be iterable premice (mice) then there exist normal iterations $\mathcal{I}$ on M with last model M^* and $\mathcal{J}$ on N with last model N^* that one of the following holds:

(i) $M^* \trianglelefteq N^*$ *and* $\mathcal{I}$ *does not truncate;*

(ii) $N^* \trianglelefteq M^*$ *and* $\mathcal{J}$ *does not truncate.*

The core model

The core model K is the minimal maximal mouse (not the actual definition!) if such exists. K exists if there is no mouse satisfying $(*)_\lambda$.

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K has the following properties (among others):

- $\mathrm{cof}((\alpha^+)^K) \geq \kappa$ for all $\alpha \geq \aleph_2$;
- if $\mathrm{cof}(\alpha) < \mathrm{Card}(\alpha)$ but $\alpha \geq \aleph_2$ is regular in K then, in fact, $\mathrm{o}^K(\alpha) \geq \nu$ where $\mathrm{cof}(\alpha) = \omega \cdot \nu$;
- $K = K^{V[G]}$ for any set generic extension.

The covering argument

Let $\alpha \geq \aleph_2$. Assume $\mathrm{cof}((\alpha^+)^K) < \alpha$. Let $X \prec (H(\theta); \in, \ldots)$ *good* (e.g. countably closed) of size $<\alpha$ but cofinal in $(\alpha^+)^K$. Let $\sigma_X : H_X \simeq X$ and $K_X := \sigma_X^{-1}{}'' [K]$.

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  - M is a mouse, so $M \in K$,
  - $(\alpha^+)^K$ is a cardinal in M,
  - M defines a surjection from α onto $(\alpha^+)^K$.

The Theorem

Theorem

Assume $2^{\aleph_0} < \aleph_\omega$ and that there is no class size mouse of $(*)_{\omega_\omega}$.

(a) *Let $A := \{n < \omega | \exists \alpha : \aleph_n = (\alpha^+)^K \vee (\aleph_n^+)^K < \aleph_{n+1}\}$. Then $\prod_{n \in A} \aleph_n / I_{<\omega}$ carries a tree-like scale that is continuous on* $\mathrm{cof}(>\omega)$.

(b) *Let $B := \{n < \omega | \forall \alpha < \aleph_n : (\alpha^+)^K < \aleph_n \wedge (\aleph_n^+) = \aleph_{n+1}\}$. Then $\prod_{n \in B} \aleph_n / I_{<\omega}$ carries an essentially tree-like scale that is continuous on* $\mathrm{cof}(>\kappa)$*, some $\kappa < \aleph_\omega$.*

A sketch

Proof.

- W.l.o.g. assume $\aleph_n = (\kappa_n^+)^K$ for $n \in A$. Let $C_n := \{\alpha < \kappa^+ | K|\alpha \prec K|\aleph_n\}$. C_n is a club.

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- For $\alpha \in C_n$ let M_α^n be the least level of K for which a (canonical) partial surjection $g_\alpha^n : \kappa_n \to \alpha$ is k_α^n-definable. (0-definable is just $\mathbf{\Sigma}_1$-definable.)

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- For $\alpha \in C_n$ let M_α^n be the least level of K for which a (canonical) partial surjection $g_\alpha^n : \kappa_n \to \alpha$ is k_α^n-definable. (0-definable is just $\mathbf{\Sigma}_1$-definable.)
- For $\alpha \in C_m$ there exists at most one $\beta \in C_n$ such that there exists a $(k_\alpha^m =) k_\beta^n$-embedding $\sigma_{\beta,\alpha} : M_\beta^n \to M_\alpha^m$ moving certain parameters correctly. (A 0-embedding is Σ_1-elementary.)

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Proof.

- Idea: Let X be good. Then for all $n^* \leq n < m \in A$ we have a $k^n_{\chi x(\aleph_n)}$-embedding from $M^n_{\chi x(\aleph_n)}$ into $M^m_{\chi x(\aleph_m)}$.

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- Let $\langle M_\alpha, i_{\alpha,\beta} : \alpha \leq \beta \leq \gamma \rangle$ be the induced iteration on K. Let β_n be the least point in the iteration such that the next point is $\geq \sigma_X^{-1}(\kappa_n)$ (if it exists, o.w. $\beta_n = \gamma$).

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- Have then $M^n_{\chi_X(\aleph_n)} = \mathrm{Ult}(M_{\beta_n}; \sigma_X \restriction K_X || \sigma_X^{-1}(\aleph_n))$. Let $\varphi_{m,n} : M^n_{\chi_X(\aleph_n)} \to M^m_{\chi_X(\aleph_m)}$, $[a, f]_{\sigma_X} \mapsto [a, i_{\beta_n,\beta_m}(f)]_{\sigma_X}$.

□

Limitations

Theorem (Gitik)

Let κ be a regular cardinal, and let E be an extender on κ of length κ^{++}. Then there exists a forcing extension $V[G]$ with a sequence $\langle \kappa_n : n < \omega \rangle$ such that $\prod_{n<\omega} \kappa_n^{++}/I_{<\omega}$ does not carry an essentially tree-like continuous scale.

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Conjecture: Let M be a model of $(*)_\lambda$. Then there exists a forcing extension $M[G]$ with a sequence $\langle \kappa_n : n < \omega \rangle$ such that $\prod_{n<\omega} \kappa_n/I_{<\omega}$ does not carry a tree-like continuous scale.

Open Questions

Question

What is the consistency strength of the existence of a sequence $\langle \kappa_n : n < \omega \rangle$ consisting of core model successor cardinals such that $\prod_{n<\omega} \kappa_n / I_{<\omega}$ does not carry a continuous tree-like scale?

Question

Is it consistent for there to exist some singular strong limit λ such that for no $\langle \kappa_n : n < \omega \rangle$ cofinal in λ, $\prod_{n<\omega} \kappa_n / I_{<\omega}$ carries a tree-like continuous scale?

Question

Is it consistent for there to exist some singular strong limit λ and some $\langle \kappa_n : n < \lambda \rangle$ such that $\prod_{n<\omega} \kappa_n / I_{<\omega}$ carries a tree-like continuous scale and true cofinality of the product is λ^{++}?

References

[1] Saharon Shelah: *PCF and infinite free subsets in an algebra*. Archive for Mathematical Logic, vol 41, Pages 321 – 359.

[2] Luis Pereira: *The PCF conjecture and large cardinals*. JSL 73(2), Pages 647 – 688, 2008.

[3] James Cummings: *Continuous tree-like scales*. Central European Journal of Mathematics, vol 8, Pages 314 – 318.

[4] Moti Gitik: *On a question of Pereira*. Archive of Mathematical Logic, vol 47, Pages 53 – 64, 2008.

[5] Dominik Adolf and Omer Ben-Neria: *untitled*. In preparation.