DIAMOND AND GCH

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κ is regular and uncountable. Jensen introduced the following principle:

$\diamondsuit_\kappa$: there exists a sequence $\langle S_\alpha : \alpha < \kappa \rangle$ such that for every $S \subseteq \kappa$, $\{\alpha : S \cap \alpha = S_\alpha\}$ is stationary.

Remark 1: using diamond, you can anticipate all 2^κ many subsets of κ in a mere κ steps.

Some historical remarks:

(1) (Jensen) If $\diamondsuit_{\omega_1}$ then there is a Souslin tree.
(2) (Jensen) If $V = L$ then $\diamondsuit_\kappa$ for all uncountable regular κ.
(3) If S is bounded then $S_\alpha = S$ for many α, so $\diamondsuit_\kappa$ implies $\kappa^{<\kappa} = \kappa$. In particular $\diamondsuit_{\omega_1}$ implies CH.
(4) (Jensen) Consistently CH plus no Souslin tree, so in particular CH does not imply $\diamondsuit$.
(5) (Gregory) If GCH holds then $\diamondsuit_{\lambda^+}$ for all regular uncountable λ.
(6) (Shelah) If GCH holds then $\diamondsuit_{\lambda^+}$ for all singular λ.

Recently Shelah proved the optimum result of this latter type: if λ is uncountable and $2^\lambda = \lambda^+$, then $\diamondsuit_{\lambda^+}$.

Remark 2: An equivalent version of $\diamondsuit_\kappa$ for functions rather than sets: there is $\langle f_\alpha : \alpha < \kappa \rangle$ such that for every $f : \kappa \to \kappa$, $f \restriction \alpha = f_\alpha$ for stationary many α.

Why is it equivalent? If you have the function version just set $S_\alpha = \{\beta : f_\alpha(\beta) \neq 0\}$. For the converse fix a bijection $g : \kappa \simeq \kappa^2$, note that for a club of α we have $g \restriction \alpha : \alpha \simeq \alpha^2$, and set $f_\alpha = g``S_\alpha$ whenever $g \restriction \alpha : \alpha \simeq \alpha^2$ and $g``S_\alpha$ is a function.

For simplicity we'll prove the following special case of Shelah's result (which has all the main ideas): if $2^{\omega_1} = \omega_2$ then $\diamondsuit_{\omega_2}$. Let $S = \omega_2 \cap cof(\omega)$: we'll build f_α for $\alpha \in S$ so that every f is guessed on a stationary subset of S.

We start by fixing

(1) A sequence $\langle f_\beta : \beta < \omega_2 \rangle$ such that
   (a) $dom(f_\beta) \leq \beta$ and $rge(f_\beta) \subseteq \omega_2$.
   (b) Every function g with $dom(g) < \omega_2$ and $rge(g) \subseteq \omega_2$ appears as f_β for unboundedly many $\beta < \omega_2$.
(2) A sequence $\langle g_i : i < \omega_1 \rangle$ such that
   (a) $g_i : \omega_2 \to \omega_2$.
   (b) For every $h : \omega_1 \to \omega_2$, there are unboundedly many $\beta < \omega_2$ such that $\forall i \; h(i) = g_i(\beta)$.
(3) For each β a sequence $\langle x_i^\beta : i < \omega_1 \rangle$ of countable subsets of such that
   (a) x_i^β increases with i.
   (b) $\beta = \bigcup_i x_i^\beta$.

We let $f^1_{\beta,i} = g_i \circ f_\beta$, and note that $dom(f^1_{\beta,i}) = dom(f_\beta)$.

Given $h : \omega_2 \to \omega_2$ and $i < \omega_1$ we define $S^{h,i}$ to be the set of $\beta \in S$ such that

(1) x_i^β is unbounded in β.

(2) For unboundedly many $\alpha \in x_i^\beta$ there is $\bar\alpha > \alpha$ such that $\bar\alpha \in x_i^\beta$ and $h \restriction \alpha = f^1_{\bar\alpha,i}$.

Intuition: if $\beta \in S^{h,i}$ then we can "assemble" $h \restriction \beta$ using only functions from the fixed countable set $\{f^1_{\gamma,i} : \gamma \in x_i^\beta\}$ (which is independent of h!)

Key claim 1: there is $i < \omega_1$ such that $S^{h,i}$ is stationary for all h.

Proof: Otherwise we may fix C_i club and h_i so that $C_i \cap S^{h_i,i} = \emptyset$. Now we let $h(\alpha)$ be the least $\beta > \alpha$ such that $\langle h_i(\alpha) : i < \omega_1\rangle = \langle g_i(\beta) : i < \omega_1\rangle$, and then $H(\alpha)$ be the least $\gamma > \alpha$ such that $h \restriction \alpha = f_\gamma$.

Let D be the club set of β which are closed under H, and find $\beta \in S \cap \bigcap_i C_i \cap D$. Let Z be cofinal in β with order type ω,and find $j < \omega_1$ large enough that $Z \cup H``Z \subseteq x_j^\beta$. We claim that $\beta \in S^{h_j,j}$, which is a contradiction since also $\beta \in C_j$.

Since $Z \subseteq x_j^\beta$, x_j^β is cofinal in β. Let $\alpha \in Z$ and let $\bar\alpha = H(\alpha)$, so that also $\bar\alpha \in x_j^\beta$. By definition $h \restriction \alpha = f_{\bar\alpha}$, so that for every $\eta < \alpha$ we have

$$f^1_{\bar\alpha,j}(\eta) = g_j(h(\eta)) = h_j(\eta).$$

So $h_j \restriction \alpha = f^1_{\bar\alpha,j}$. This shows that $\beta \in S^{h_j,j}$, concluding the proof of key claim 1.

We fix i as in key claim 1, and let $f^2_{\alpha,j} = g_j \circ f^1_{\alpha,i}$ for all j and α. For $\beta \in S$ let v_β be the set of pairs $(\alpha,\bar\alpha)$ such that $\alpha, \bar\alpha \in x_i^\beta$ and $dom(f_{\bar\alpha}) = \alpha$. Note that this is a countable set.

We now aim to find $w_\beta \subseteq v_\beta$ and j such that for many β, $\bigcup_{(\alpha,\bar\alpha)\in w_\beta} f^2_{\bar\alpha,j}$ is a function, and such that these functions form a diamond sequence.

The details: we will choose by recursion (for as long as we can) E_j, h_j for $j < \omega_1$ such that

(1) The E_j are club subsets of ω_2 decreasing with j, and $E_k = \bigcap_{k<j} E_k$ for k limit.
(2) The h_j are functions from ω_2 to ω_2.
(3) If we define $v_{\beta,j}$ for $\beta \in E_j \cap S$ to be the set of $(\alpha,\bar\alpha) \in v_\beta$ such that $h_k \restriction \alpha = f^2_{\bar\alpha,k}$ for all $k < j$, then for every $\beta \in E_{j+1} \cap S$ EITHER $\{\alpha : \exists\bar\alpha\ (\alpha,\bar\alpha) \in v_{\beta,j}\}$ is bounded in β OR $v_{\beta,j+1} \subsetneq v_{\beta,j}$.

To start let $E_0 = \omega_2$. Suppose that we have chosen E_k and h_k for $k < j$ (note that if j is limit then we are forced to define E_j as the intersection of the preceding E_k's). We compute the sequence $\langle v_{\beta,j} : \beta \in E_j \cap S\rangle$ and then ask the following question: is there a function h and a club $E \subseteq E_j$ such that for every $\beta \in E \cap S$ either $v_{\beta,j}$ is bounded in β or $\{(\alpha,\bar\alpha) \in v_{\beta,j} : h \restriction \alpha = f^2_{\bar\alpha,j}\} \subsetneq v_{\beta,j}$? If yes then we choose $h_j = h$ and $E_j = E$. Otherwise we halt and set $w_\beta = v_{\beta,j}$ for $\beta \in E_j \cap S$.

Key claim 2: We halt before ω_1 steps.

Proof: suppose for contradiction that h_j, E_j are defined for all $j < \omega_1$. Let $E_\infty = \bigcap_j E_j$, and let $h(\alpha)$ be the least $\beta > \alpha$ such that $\langle h_j(\alpha)\rangle = \langle g_j(\beta)\rangle$. Now recall that $S^{h,i}$ is stationary, and choose $\beta \in E_\infty \cap S^{h,i}$. By definition x_i^β is unbounded in β and there are unboundedly many $\alpha \in x_i^\beta$ such that for some $\bar\alpha \in x_i^\beta$ we have $h \restriction \alpha = f^1_{\bar\alpha,i}$

Let $\alpha, \bar\alpha \in x_i^\beta$ be such that $h \restriction \alpha = f^1_{\bar\alpha,i}$. Then by definition of h, for every $j < \omega_1$ and every $\eta < \alpha$

$$f^2_{\bar\alpha,j}(\eta) = g_j(f^1_{\bar\alpha,i}(\eta)) = g_j(h(\eta)) = h_j(\eta),$$

that is $h_j \restriction \alpha = f^2_{\bar\alpha,j}$. So $(\alpha,\bar\alpha) \in v^\beta_j$ for all $j < \omega_1$. It follows that for all j the set $\{\alpha : \exists\bar\alpha\ (\alpha,\bar\alpha) \in v_{\beta,j}\}$ is unbounded in β, and since $\beta \in E_{j+1}$ we have $v_{\beta,j+1} \subsetneq v_{\beta,j}$. But this is impossible because v_β is countable, concluding the proof of key claim 2.

Suppose the construction halts at stage j, that is we are unable to make a choice for h_j and E_{j+1}. Recall that we set $w_\beta = v_{\beta,j}$ for $\beta \in E_j \cap S$.

Key claim 3: For every $f : \omega_2 \to \omega_2$ there are stationarily many $\beta \in E_j \cap S$ such that

1. $\{\alpha : \exists\bar\alpha\ (\alpha,\bar\alpha) \in w_\beta\}$ is unbounded in β.
2. $\bigcup_{(\alpha,\bar\alpha)\in w_\beta} f^2_{\alpha,j}$ is a function with domain β.
3. $f \restriction \beta = \bigcup_{(\alpha,\bar\alpha)\in w_\beta} f^2_{\alpha,j}$.

Proof: Suppose not and fix a club set $E \subseteq E_j$ such that for every $\beta \in E \cap S$ one of the statements above is false. We claim that we could have set $h_j = f$ and $E_{j+1} = E$, contradicting the choice of j as the stage where the construction halted. Note that if we set $h_j = f$ then for every $\beta \in E_{j+1} \cap S$ we have $v_{\beta,j} = w_\beta$, $v_{\beta,j+1} = \{(\alpha,\bar\alpha) \in w_\beta : f \restriction \alpha = f^2_{\bar\alpha,j}\}$.

For every $\beta \in E_{j+1} \cap S$ one of the following occurs:

1. $\{\alpha : \exists\bar\alpha\ (\alpha,\bar\alpha) \in v_{\beta,j}\}$ is bounded in β.
2. $\{\alpha : \exists\bar\alpha\ (\alpha,\bar\alpha) \in v_{\beta,j}\}$ is unbounded in β, and there are $(\alpha_i,\bar\alpha_i) \in v_{\beta,j}$ for $i = 0,1$ such that the functions $f^2_{\bar\alpha_i,j}$ are incompatible: in this case one of the pairs $(\alpha_i,\bar\alpha_i)$ is not in $v_{\beta,j+1}$ because $f \restriction \beta$ can't agree with both the functions $f^2_{\bar\alpha_i,j}$, so that $v_{\beta,j+1} \subsetneq v_{\beta,j}$.
3. $\{\alpha : \exists\bar\alpha\ (\alpha,\bar\alpha) \in v_{\beta,j}\}$ is unbounded in β, and $\bigcup_{(\alpha,\bar\alpha)\in v_{\beta,j}} f^2_{\alpha,j}$ is a function with domain β which is not equal to $f \restriction \beta$: in this case there is $(\alpha.\bar\alpha) \in v_{\beta,j}$ such that $f \restriction \alpha \neq f^2_{\bar\alpha,j}$, and so $(\alpha,\bar\alpha) \notin v^\beta_{j+1}$ and again $v_{\beta,j+1} \subsetneq v_{\beta,j}$.

It follows that we could continue the construction, contradicting the choice of j.

Conclusion: If we define $f^3_\beta = \bigcup_{(\alpha,\bar\alpha)\in w_\beta} f^2_{\alpha,j}$ whenever this defines a function with domain β, then we have a diamond sequence.