The Regularity Lemma IV: Separable realisations

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Recall:

- The *Regularity Lemma (for graphs)* says very roughly that the vertices of a large enough graph can be partitioned into not too many chunks of roughly equal size, so that between most pairs of chunks the edges look random.

- We outlined a proof (due to Elek and Szegedy): again very roughly the idea is
  1. Start with a counterexample in the form of an infinite sequence of finite graphs $G_n$.
  2. Take an *ultraproduct* to form an uncountable graph $G_\infty$ together with measures on its vertex and edge sets.
  3. Use ideas from measure theory to prove a version of regularity for $G_\infty$, and then “pull back” to show that some $G_n$ is “too regular”.
An non-principal ultrafilter on $\mathbb{N}$ is a finitely additive non-atomic 0 − 1 measure on $P(\mathbb{N})$; that is to say a function \( \mu : P(\mathbb{N}) \to \{0, 1\} \) such that

1. \( \mu(A \cup B) = \mu(A) + \mu(B) \) for \( A, B \) disjoint.
2. \( \mu(\{n\}) = 0 \) for all \( n \in \mathbb{N} \).
3. \( \mu(\mathbb{N}) = 1 \).

Such objects exist but need the axiom of choice to construct them. We’ll fix for the rest of this talk such a measure \( \mu \), and let \( U = \{ A : \mu(A) = 1 \} \).

If \( (r_n) \) is a bounded sequence of reals we’ll let \( \lim_\mu r_n \) be the unique real \( r \) such that for every \( \epsilon > 0 \), \( r - \epsilon < r_n < r + \epsilon \) for \( \mu \)-ae \( n \).
Let \((X_n)\) be finite sets such that \(|X_n|\) is strictly increasing. Then we defined the \textit{ultraproduct} \(X_\infty\) as follows: elements are represented by functions \(f\) such that \(f(n) \in X_n\) for \(\mu\)-ae \(n\), and two such functions are identified if they agree \(\mu\)-ae.

Given sets \((A_n)\) with \(A_n \subseteq X_n\), in the natural way we have \(A_\infty \subseteq X_\infty\). We define a measure on such sets by \(\nu(A_\infty) = \lim \mu |A_n|/|X_n|\). The sets of form \(A_\infty\) are a ring of sets.

We extend this measure as follows: say that an arbitrary \(B \subseteq X_\infty\) is \textit{null} if for all \(\epsilon > 0\) there is a set of form \(A_\infty\) with \(B \subseteq A_\infty\) and \(\nu(A_\infty) < \epsilon\).

Let \(B\) be the set of subsets of \(X_\infty\) which differ from a set of form \(A_\infty\) by a null set. Extend \(\nu\) to \(B\) in the obvious way. Then \(B\) is a \(\sigma\)-algebra and \(\nu\) is a countably additive measure.
We also need “two dimensional” versions of these notions. Starting with counting measure on the sets $X_n^2$, we obtain a $\sigma$-algebra $\mathcal{B}_2$ on $X_\infty^2$ and a countably additive measure living on $\mathcal{B}_2$.

The following points are critical for the success of the argument (we’ll see sketchy proofs of the first two assertions later, the third follows from “finite Fubini” and a limiting argument).

- The measure spaces $(X_\infty, \mathcal{B}, \nu)$ and $(X_\infty^2, \mathcal{B}_2, \nu_2)$ are not separable.
- $\nu_2$ is not the product measure obtained from two copies of $\nu$.
- A version of Fubini’s theorem holds for $\nu_2$ and $\nu$. 
The key lemma in the Elek-Szegedy proof is that there exists many *separable realisations*. Before defining this concept we review some measure theory.

1. **A probability measure space** is a triple \((X, \mathcal{A}, \mu)\) where \(\mathcal{A}\) is a \(\sigma\)-algebra on \(X\) and \(\mu\) is a countably additive measure defined on \(\mathcal{A}\) with \(\mu(X) = 1\).

2. If \((X, \mathcal{A}, \mu)\) is such a space the associated *measure algebra* is the algebra obtained by quotienting by the null sets. The measure algebra can be made into a complete metric space by setting \(d([A], [B]) = \mu(A, B)\).

3. The measure space \((X, \mathcal{A}, \mu)\) is *separable* iff the associated metric space is separable, that is it has a countable dense subset.
By a (very special case of a) theorem of Maharam there is essentially only one separable atomless probability measure space, namely $([0, 1], \mathcal{B}, \lambda)$ where $\mathcal{B}$ is the algebra of Lebesgue measurable sets and $\lambda$ is Lebesgue measure.

(Maharam) If $(X, \mathcal{A}, \mu)$ is a separable atomless probability measure space there exists $f : X \rightarrow [0, 1]$ such that

1. $f^{-1}(\mathcal{B}) \subseteq \mathcal{A}$.
2. $\mu(f^{-1}(U)) = \lambda(U)$ for all $U \in \mathcal{B}$.
3. Every $W \in \mathcal{A}$ is equal modulo null sets to a set $f^{-1}(U)$ for some $U \in \mathcal{B}$.

We’ll say that such an $f$ is a Maharam function for the space $(X, \mathcal{A}, \mu)$. 

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Let \((X, \mathcal{A}, \mu)\) be a measure space and let \(\mathcal{A}_0, \mathcal{A}_1\) be subalgebras of \(\mathcal{A}\). Then

1. \(\mathcal{A}_0\) and \(\mathcal{A}_1\) are independent iff \(\mu(B_0 \cap B_1) = \mu(B_0) \times \mu(B_1)\) for \(B_i \in \mathcal{A}_i\).

2. \(\mathcal{A}_1\) is a independent complement for \(\mathcal{A}_0\) if they are independent and \(\mathcal{A}_0 \cup \mathcal{A}_1\) generates \(\mathcal{A}\).

Let \(\mathcal{B}^*\) be the subalgebra of \(\mathcal{B}\) generated by all “rectangles” of form \(X \times Y\) where \(X, Y \in \mathcal{B}\). It is easy to see that \(\mathcal{B}^* \subseteq \mathcal{B}_2\) and that \(\nu_2(X \times Y) = \nu(X)\nu(Y)\).
Let $E \in \mathcal{G}_2$. A separable realisation for $E$ consists of separable subalgebras $\sigma_1 \subseteq \mathcal{B}$ and $\sigma_2 \subseteq \mathcal{B}_2$, together with Maharam functions $f_1$ and $f_2$ for the corresponding spaces. These must satisfy:

1. $E$ is in the $\sigma$-algebra $\tau$ generated by $\sigma_x$, $\sigma_y$ and $\sigma_2$, where $\sigma_x = \{A \times X_\infty : A \in \sigma_1\}$ and $\sigma_y = \{X_\infty \times A : A \in \sigma_1\}$.
2. (CRUCIAL FOR REGULARITY) $\sigma_2$ is independent of $\mathcal{B}^*$. 
3. $\sigma_2$ consists of sets $A$ which are symmetric in the sense that $(a, b) \in A \iff (b, a) \in A$.
4. $f_2$ is symmetric in the sense that $f_2(a, b) = f_2(b, a)$.

Note that by independence the map $(x, y) \mapsto (f_1(x), f_1(y), f_2(x, y))$ is an equivalence between $(X^2_\infty, \tau, \nu_2)$ and the Lebesgue measure algebra for $[0, 1]^3$. 

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To build separable realisation we need some way of getting independent complements. Recall the idea of *conditional expectation*:

If \((X, \mathcal{A}, \mu)\) is a measure space, \(\mathcal{B}\) is a subalgebra of \(\mathcal{A}\) and \(f\) is an integrable \(\mathcal{A}\)-measurable function then \(E(f|\mathcal{B})\) is the unique (modulo nullsets) integrable \(\mathcal{B}\)-measurable function \(g\) such that

\[
\int_B f d\mu = \int_B g d\mu
\]

for all \(B \in \mathcal{B}\).

If \(A \in \mathcal{A}\) we abuse notation and write \(E(A|\mathcal{B})\) where \(A\) is shorthand for the characteristic function \(\chi_A\).
Let \((X, \mathcal{A}, \mu)\) be a probability measure space and let \(\mathcal{B}\) be a subalgebra of \(\mathcal{A}\). Then a \(\mathcal{B}\)-random \(k\)-partition in \(\mathcal{A}\) is a partition of \(X\) into sets \(A_1, \ldots A_k \in \mathcal{A}\) such that \(E(A_i|\mathcal{B}) = 1/k\) for all \(i\).

Intuition: the \(A_i\) are candidates to live in a subalgebra which is independent of \(\mathcal{B}\).

Technical result: Let \((X, \mathcal{A}, \mu)\) be a separable probability measure space and let \(\mathcal{B}\) be a subalgebra of \(\mathcal{A}\), such that for every \(k\) there is a \(\mathcal{B}\)-random \(k\)-partition in \(\mathcal{A}\). Then there is an independent complement for \(\mathcal{B}\) in \(\mathcal{A}\).
Sketch of proof:

1. Build a tree of sets in $A$, such that
   1. Each level is a finite partition of $X$.
   2. The sets in the tree generate $A$.
   3. If $R$ is a set on level $k$ then $E(R|B) \leq 1/k$.

2. Construct from the tree sets $S(\lambda) \in A$ such that $E(S(\lambda)|B) = \lambda$. It is easy to see that if $C$ is the algebra generated by the $S(\lambda)$’s then $C$ is independent of $B$.

3. Argue that $B \cap C$ generates $A$, using the fact that $S(\lambda)$’s were constructed from a generating set.
Now for the central technical lemma: recall that we had a sequence of finite sets \((X_n)\), we formed the ultraproduct \(X_\infty\), and that we defined measure spaces \((X_\infty, \mathcal{B}, \nu)\) and \((X^2_\infty, \mathcal{B}_2, \nu_2)\) essentially as limits of the counting measures on \(X_n\) and \(X^2_n\).

Main Lemma: for every \(k\) there is a \(\mathcal{B}^*\)-random \(k\)-partition in \(\mathcal{B}_2\).
We need a couple of basic results in probability.

Borel-Cantelli: if $E_n$ are events such that $\sum_n p(E_n) < \infty$, then the probability that infinitely many occur is zero.

Chernoff bound (a crude special case): Let $k$ be a positive integer and let $\epsilon > 0$. Let $\Omega_N$ be the finite probability space where the points are $N$-tuples from $\{1, \ldots, k\}$ each with probability $k^{-N}$ (think of tossing a fair $k$-sided die $N$ times).

There is a constant $c = c(k, \epsilon) > 0$ such that for all $N$ and all $i$,

$$P(\{|j : s(j) = i| \notin ((1/k - \epsilon)N, (1/k + \epsilon)N)\} < e^{-cN}$$
Proof of Main Lemma: Choose at random for each $n$ a partition $F_n : X_n^2 \to \{1, \ldots k\}$. Then partition $X_\infty^2$ by the rule that $F_\infty([f])$ is the unique $i$ such that $F_n(f(n)) = i$ for $\mu$-ae $n$. This works.

Details: Think of the sequence $(F_n)$ as function from $Z = \bigcup_n X_n^2$ to $\{1, \ldots k\}$. Give the space $\Omega$ of all such functions Bernoulli measure: a *cylinder* is given by prescribing the values of the function on a finite set $S \subseteq Z$, and the measure of such a cylinder is $k^{-|S|}$.

For each $n$ let $\mathcal{E}_n$ be the family of subsets of $X_n$ of form $A \times B$ where $A, B \subseteq X_n$. Note that

1. The algebra $\mathcal{B}^*$ is generated by sets represented in the ultrapower by sequences $(E_n)$, with $E_n \in \mathcal{E}_n$.
2. The family $\mathcal{E}_n$ has size $2^{2|X_n|}$. This is *much less* than $2^{2|X_n|^2}$.
Let $0 < \epsilon < (10k)^{-1}$, and let $E \in \mathcal{E}_n$ be such that $|E| \geq \epsilon |X_n|^2$.

By the Chernoff inequality, the probability that $f \in \Omega$ takes the value 1 (say) on $E$ more than $(1/k + \epsilon)|E|$ or less than $(1/k - \epsilon)|E|$ times (we’ll say “$f$ deviates by $\epsilon$ on $E$ at $n$”) is bounded by $e^{-c\epsilon|E|}$.

So the probability that there is $E$ as above such that $f$ deviates by $\epsilon$ on $E$ at $n$ is bounded by $p_n = 2^{2|X_n|} 2e^{-c\epsilon|X_n|^2}$. Since $|X_n|$ is strictly increasing, $\sum_n p_n$ converges so by Borel-Cantelli we get that

For ae $f \in \Omega$, there are only finitely many $n$ such that for some $E$ as above, $f$ deviates by $\epsilon$ on $E$ at $n$. By countable additivity, for ae $f \in \Omega$ for every $\epsilon > 0$ there are only finitely many $n$ such that $f$ deviates by $\epsilon$ on some $E$ with $|E| \geq \epsilon |X_n|^2$ at $n$. Call such $f$ good.
Given $f$ good, let $S_1^f$ be cell 1 in the induced partition of $X_\infty^2$. Consider a typical generating set $E^*$ for $\mathcal{B}_2$, represented by a sequence $(E_n)$ with $E_n \in \mathcal{E}_n$, we claim that $\nu_2(S_1^f \cap E^*) = \nu_2(E^*) / k$. If $\nu_2(E^*) = 0$ there is nothing to do, so let $\nu_2(E^*) = \gamma > 0$ and choose $\epsilon$ positive and much smaller than $\gamma$.

By definition of $\nu_2$ as a limit of counting measure, there is a $\mu$-large set of $n$ such that $|E_n|/|X_n|^2$ is within $\epsilon$ of $\gamma$ (so in particular $|E_n| > \epsilon|X_n|^2$). As $f$ is good among these $n$ there are only finitely many where $f$ deviates by $\epsilon$ on $E_n$, and finite sets have $\mu$-measure zero, so (using the definition of $\nu_2$ again) $\nu_2(S_1^f \cap E^*)$ is between $(1/k - \epsilon)\nu_2(E^*)$ and $(1/k + \epsilon)\nu_2(E^*)$.

It follows easily that $\nu_2(S_1^f \cap F) = \nu_2(F)/k$ for every $F \in \mathcal{B}^*$.